Fractal properties of spacing distributions

1997 J. Phys. A: Math. Gen. 30 915

(http://iopscience.iop.org/0305-4470/30/3/016)

J. Phys. A: Math. Gen. **30** (1997) 915–926. Printed in the UK PII: S0305-4470(97)73427-9

Fractal properties of spacing distributions

Chris Greenman
Department of Mathematics, Bolton Institute, Deane Road, Bolton BL3 5AB, UK

Received 9 April 1996

Abstract. The paper reports a link between the Hausdorff dimension of a number theoretically based set, and certain arithmetic properties of the spacing distribution of the two-dimensional harmonic oscillator. It is shown that the set of points $\omega \in [0, 1]$, with continued fraction $[a_1, a_2, \ldots]$, such that $\log \prod_{i=1}^{n} a_i^{1/n}$ diverges, has Hausdorff dimension $\frac{1}{2}$. The set of convergents $p_n/q_n = [a_1, \ldots, a_n]$, such that the series $q_n^{1/n}$ diverge, is also shown to have a Hausdorff dimension $\frac{1}{2}$. Although this result can be seen as a purely number-theoretic result, it is related to level spacing distributions in the following manner. For the two-dimensional harmonic oscillator with frequency ratio, ω, that has a continued fraction satisfying the above condition, the level spacing distribution is $\delta(s)$. Thus, the non-ergodic behaviour of the two-dimensional oscillator has Hausdorff dimension $\frac{1}{2}$. Similar results are found for the system of a particle trapped in a box, using a number-theoretic result of Ramanujan.

1. Introduction

This paper investigates the Hausdorff dimension of two sets arising in continued fraction theory. Continued fractions have many applications in the Diophantine approximation. A multitude of examples can be found in [1] or [2] for example. For other results on Hausdorff dimensions relating to continued fractions, [3] is a good source. Any number $\omega \in [0, 1]$ can be expressed uniquely as a continued fraction,

$$\omega = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \tag{1}$$

where the a_i are positive integers. The convergents are defined by,

$$\frac{p_n}{q_n} = [a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{a_n}}} \tag{2}$$

which are alternately upper and lower bounds for ω as $n \to \infty$. The following induction laws are obeyed,

$$\begin{array}{lll} p_1 = 1 & p_2 = a_2 & p_n = a_n p_{n-1} + p_{n-2} \\ q_1 = a_1 & q_2 = a_1 a_2 + 1 & q_n = a_n q_{n-1} + q_{n-2}. \end{array} \tag{3}$$

The Hausdorff dimension is a measure of a set size, giving fractional dimensions for sets of fractal nature, such as the Cantor set. Examples can be found in [3] and [4]. The

0305-4470/97/030915+12$19.50

Hausdorff measure of any set, X, arises as follows. First an ϵ-cover of X by balls A_i is formed, where diam $A_i \leqslant \epsilon$. Then define the function

$$H^s_\epsilon(X) = \inf\left\{\sum_i (\mathrm{Diam}\, A_i)^s\right\} \tag{4}$$

where the infimum is taken over all possible ϵ-covers of X. The Hausdorff measure is then defined as

$$H^s(X) = \lim_{\epsilon\to 0} H^s_\epsilon(X).$$

It can then be shown that,

$$H^s(X) = \begin{cases} \infty & s < D(X) \\ 0 & s > D(X) \end{cases} \tag{5}$$

there $D(X)$ is the Hausdorff dimension of X. This paper proves the following results.

Theorem 1. The set of points $\omega \in [0,\ 1]$, with continued fraction $[a_1, a_2, \ldots]$, such that

$$\lim_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} a_i = \infty \tag{6}$$

has the Hausdorff dimension $\frac{1}{2}$.

Corollary 1.1. For convergents $\frac{p_n}{q_n} = [a_1, \ldots, a_n]$, such that the series $q_n^{\frac{1}{n}}$ diverge, the Hausdorff dimension is $\frac{1}{2}$.

The paper will proceed as follows. In section 2 the method of the proof will be outlined in more detail. Section 3 gives an arithmetical bound, which is used to obtain an upper bound of $\frac{1}{2}$ for the Hausdorff dimension of theorem 1. Section 5 uses the measure theory arguments to obtain a lower bound of $\frac{1}{2}$, which completes theorem 1. In section 6, the corollary is proved. Conclusions follow in the final section, where the particle in a box is examined.

Note that both sets being measured are dense in $[0, 1]$. To see this, note that any number $[b_1, b_2, \ldots]$ can be approximated arbitrarily closely by $[b_1, \ldots, b_n, c_1, c_2, \ldots]$, where $c_i = \mathrm{i}!$, which is a member of the set measured in theorem 1 and in corollary 1.1.

It is also worth remarking that these studies arose from work on the quantum mechanics of dynamical systems. The level spacing distribution of the two-dimensional harmonic oscillator was studied, where it was found that if ω is the ratio of the frequencies of the oscillations, such that the conditions of theorem 1 were satisfied, $\delta(s)$ was obtained. Further details can be found in [7] and [8].

2. The method of proof

Let X be the set of points $\omega \in [0,\ 1]$ that satisfy (6). The first half of the proof will show that $\frac{1}{2}$ is an upper bound of the Hausdorff dimension, denoted $D(X)$. A covering set $X(\kappa)$ of X is defined by $[a_1, a_2, \ldots] \in X(\kappa)$ if and only if,

$$\lim_{n\to\infty} \frac{1}{n} \log\left[\prod_{i=1}^{n} a_i\right] > \log \kappa$$

where κ is finite. So $X(\kappa)$ covers X with $\lim_{\kappa\to\infty} X(\kappa) = X$. So if $[a_1, a_2, \ldots] \in X(\kappa)$, then for a suitably large N, $\forall n > N$,

$$\prod_{i=1}^{n} a_i > \kappa^n. \tag{7}$$

With this, $D(X(\kappa))$ will be shown to have an upper bound of $\frac{1}{2}$. As $D(X) \leqslant D(X(\kappa))$ it then follows that $D(X)$ will have an upper bound of $\frac{1}{2}$. To obtain a lower bound of $\frac{1}{2}$, the following argument is used. First the subset $Y \subset X$ is defined as follows. Let $(a_1, a_2, \ldots, a_n)$ denote the set of points that have continued fraction $[a_1, \ldots a_n, \ldots]$, where $a_1, \ldots, a_n$ are fixed and $a_{n+1}, a_{n+2}, \ldots$ are free ranging over the natual numbers. Thus $(a_1, a_2, \ldots, a_n) = [a_1, a_2, \ldots, a_{n-1}, a_n + t]$ is the interval covered as t ranges over $[0, 1)$. Then define,

$$Y_n = \bigcup_{a_1=1}^{4} \bigcup_{a_2=2}^{7} \ldots \bigcup_{a_n=n}^{3n+1} (a_1, \ldots, a_n). \tag{8}$$

Then the set Y arises as the intersection of the nested sets Y_n, i.e.

$$Y = \bigcap_{n=1}^{\infty} Y_n = \lim_{n\to\infty} Y_n.$$

To see that this is a subset of X, note that all points in Y_n require $a_i \geqslant$ i, so $\prod_{i=1}^{n} a_i \geqslant n!$ and $\log \prod_{i=1}^{n} a_i \geqslant n \log n$ (by Stirling's formula, which can be found in [9]), and so

$$\lim_{n\to\infty} \frac{\log(a_1 \ldots a_n)}{n} = \infty$$

as required by (6). If a measure, μ, is found such that for any interval, I, of any ϵ-cover $\bar{Y}$ of Y,

$$\mu(I) \leqslant A|I|^{\frac{1}{2}} \tag{9}$$

for any constant, $A > 0$, then

$$\mu(\bar{Y}) \leqslant A \sum_{I\in\bar{Y}} |I|^{\frac{1}{2}}.$$

As this is true for any cover, $\bar{Y}$, the infimum may be taken to give,

$$\mu(\bar{Y}) \leqslant A H_{\epsilon}^{\frac{1}{2}}(Y).$$

So as $Y \subset \bar{Y}$, $\mu(Y) < \mu(\bar{Y})$ and

$$H_{\epsilon}^{\frac{1}{2}}(Y) \geqslant \frac{\mu(Y)}{A}.$$

The final step is to show that $\mu(Y) > 0$, which yields

$$H_{\epsilon}^{\frac{1}{2}}(Y) > 0 \tag{10}$$

which in comparison with (5) gives,

$$D(Y) > \tfrac{1}{2}.$$

Noting that $Y \subset X$, it follows that $D(Y) < D(X)$ and the proof of theorem 1 is complete. The following section obtains the number theoretical bound necessary to obtain the upper bound on the Hausdorff dimension.

3. The applicable arithmetic number theory

If $d_n(m)$ denotes the number of ways of expressing m as the product of n positive factors (any number of which may be unity), where only the order of the factors is to be regarded as distinct, then,

Lemma 3.1.

$$\sum_{m>x} \frac{d_n(m)}{m^r} \leqslant \left(\frac{2\zeta(r)}{r-1}\right)^n \frac{1}{x^{r-1}} \sum_{l=0}^{n-1} \frac{(\log x)^l}{l!} \tag{11}$$

where $1 \leqslant r \leqslant 2$.

The application of this lemma arises by putting $r = 2s$, at which point the sum on the left-hand side becomes the sum in (4) used to obtain the upper bound on the Hausdorff dimension. Note that the lemma can be extended to $r \geqslant 2$ by simply removing the $r-1$ denominator, although this will have no implication on the Hausdorff dimension (which is certainly not greater than unity).

Proof. The method of proof used will be that of induction. First define,

$$c(n,x) = \sum_{\{a_1\ldots a_n \geqslant x\}} \prod_{i=1}^{n} a_i^{-r} = \sum_{m>x} \frac{d_n(m)}{m^r}. \tag{12}$$

Now assume that (11) is true for n. Then with the observation that,

$$\sum_{a_1\ldots a_n} \prod_{i=1}^{n} a_i^{-r} = \zeta(r)^n$$

the following recursive formula can be formed,

$$c(n+1,x) = \sum_{k=1}^{[x]} k^{-r} c\left(n, \frac{x}{k}\right) + \zeta(r)^n \sum_{[x]+1}^{\infty} k^{-r}.$$

So using the inductive hypothesis,

$$\begin{aligned} c(n+1,x) &\leqslant \left(\frac{2\zeta(r)}{r-1}\right)^n \sum_{k=1}^{[x]} k^{-r} \left(\frac{k}{x}\right)^{r-1} \sum_{l=0}^{n-1} \frac{\left(\log \frac{x}{k}\right)^l}{l!} + \zeta(r)^n \sum_{[x]+1}^{\infty} k^{-r} \\ &= \left(\frac{2\zeta(r)}{r-1}\right)^n \frac{1}{x^{r-1}} \sum_{l=0}^{n-1} \frac{1}{l!} \sum_{k=1}^{[x]} \frac{\left(\log \frac{x}{k}\right)^l}{k} + \zeta(r)^n \sum_{[x]+1}^{\infty} k^{-r}. \end{aligned}$$

Then first,

$$\sum_{[x]+1}^{\infty} k^{-r} \leqslant |k^{-r}|_{k=x} + \int_x^{\infty} \frac{\mathrm{d}k}{k^r} = \frac{1}{x^r} + \frac{1}{(r-1)x^{r-1}} \leqslant \frac{2}{(r-1)x^{r-1}}$$

so,

$$\zeta(r)^n \sum_{[x]+1}^{\infty} k^{-r} \leqslant \frac{2\zeta(r)^n}{(r-1)x^{r-1}} \leqslant \left(\frac{2\zeta(r)}{r-1}\right)^n \frac{1}{x^{r-1}}.$$

Secondly, there are bounds,

$$\sum_{k=1}^{[x]} \frac{\left(\log \frac{x}{k}\right)^l}{k} \leqslant \left|\frac{\left(\log \frac{x}{k}\right)^l}{k}\right|_{k=1} + \int_1^x \frac{\left(\log \frac{x}{k}\right)^l}{k} \mathrm{d}k = (\log x)^l + \frac{(\log x)^{l+1}}{l+1}.$$

Then putting these two bounds together gives,

$$c(n+1,x) \leqslant \left(\frac{2\zeta(r)}{r-1}\right)^n \frac{1}{x^{r-1}} \left\{ \sum_{l=0}^{n-1} \left[\frac{(\log x)^l}{l!} = \frac{(\log x)^{l+1}}{(l+1)!} \right] + 1 \right\}$$
$$\leqslant \left(\frac{2\zeta(r)}{r-1}\right)^{n+1} \frac{1}{x^{r-1}} \sum_{l=0}^{n-1} \frac{(\log x)^l}{l!}.$$

Then noting that,

$$c(1,x) = \sum_{a \geqslant x} a^{-r} \leqslant \frac{2}{(r-1)x^{r-1}} \leqslant \frac{2\zeta(r)}{(r-1)x^{r-1}}$$

completes the induction and also the proof.

4. The upper bound

As outlined in section 2, the objective here is to obtain an upper bound on $D(X(\kappa))$. To obtain the upper bound, any cover of $X(\kappa)$ can be used and considered as the infimum in (4).

Now any point $[a_1, a_2, \ldots]$ in the set $X(\kappa)$ is covered by the interval $(a_1, a_2, \ldots, a_n)$ (for any n). These intervals will make up the ϵ-covers of $X(\kappa)$, so their length is required. Each interval $(a_1, a_2, \ldots, a_n)$ can be parametrized as $l(t) = [a_1, a_2, \ldots, a_n + t]$, with $t \in [0, 1]$, where the total length L is,

$$L = \int_0^1 \left| \frac{\mathrm{d}l(t)}{\mathrm{d}t} \right| \mathrm{d}t.$$

But,

$$\frac{\mathrm{d}l(t)}{\mathrm{d}t} = -[a_1, \ldots, a_n + t]^2 \frac{\mathrm{d}}{\mathrm{d}t}[a_2, \ldots, a_n + t]$$
$$= (-1)^n \prod_{i=1}^{n} [a_i, \ldots . a_n + t]^2$$

thus,

$$L = \int_0^1 \prod_{i=1}^{n} [a_i, \ldots, a_n + t]^2 \mathrm{d}t.$$

However, there exists bounds,

$$\frac{1}{a_i} \geqslant [a_i, \ldots, a_n + t] \geqslant \frac{1}{a_i + 1} \tag{13}$$

giving,

$$\prod_{i=1}^{n} \frac{1}{a_i^2} \geqslant L \geqslant \prod_{i=1}^{n} \frac{1}{(a_i+1)^2}. \tag{14}$$

Thus, any points $[a_1, a_2, \ldots] \in X(\kappa)$ can be covered by intervals with a diameter below $\prod_{i=1}^n a_i^{-2}$, which tends to zero in the limit of n going to infinity. Now for $n > N$ (where N is defined as in (7)),

$$H_n^s(X(\kappa)) \leqslant \sum_{\{a_1,\ldots,a_n | a_1 \ldots a_n > \kappa^n\}} \prod_{i=1}^{n} \frac{1}{a_i^{2s}}.$$

Note that the subscript ϵ in the Hausdorff function has been replaced by n. The limit $\epsilon \to 0$ that gives the Hausdorff measure will then become the limit $n \to \infty$. This is where lemma 3.1 is then useful, as,

$$H_n^s(X(\kappa)) \leqslant \sum_{m>\kappa^n} \frac{d^n(m)}{m^{2s}} = c(n, \kappa^n)$$

where $2s = r$. So (11) gives the bound,

$$H_n^s(X(\kappa)) \leqslant \left(\frac{2\zeta(2s)}{2s-1}\right)^n \frac{1}{\kappa^{(2s-1)n}} \sum_{l=0}^{n-1} \frac{(\log \kappa^n)^l}{l!}.$$

Now providing κ is reasonably large ($> e$), the sum is an increasing series. Thus, replacing each term in the sum by the final term gives the bound,

$$H_n^s(X(\kappa)) \leqslant \left(\frac{2\zeta(2s)}{2s-1}\right)^n \frac{1}{\kappa^{(2s-1)n}} \frac{n^2(n\log\kappa)^n}{n!}$$

which with Stirling's formula; $n! \approx \sqrt{2\pi n} n^n \mathrm{e}^{-n}$ (which can be found in [9]), yields;

$$H_n^s(X(\kappa)) \leqslant \frac{n^{\frac{3}{2}}}{\sqrt{2\pi}} \left(\frac{2e\zeta(2s)\log\kappa}{(2s-1)\kappa^{2s-1}}\right)^n. \tag{15}$$

To find the function $H^s(X(\kappa))$, take the limit $n \to \infty$, and zero is obtained provided,

$$\frac{2e\zeta(2s)\log\kappa}{(2s-1)\kappa^{2s-1}} < 1$$

which is satisfied for any $s > \frac{1}{2}$ provided that κ is sufficiently large. Thus the Hausdorff measure is,

$$H^s(X(\kappa)) = 0$$

which in comparison with (5), gives $\frac{1}{2}$ as an upper bound on the Hausdorff dimension $D(X)$.

5. The lower bound

As outlined in section 2, a suitable measure, μ, is required on $[0, 1]$. First define the measure, μ_n, of any set $A \subseteq [0, 1]$ as,

$$\mu_n(A) = (\log 3)^{1-n} \prod_{i=1}^{n-1} \sum_{a_i=i}^{3i+1} \delta_i(a_1, \ldots, a_{n-1}) \log\left(\frac{a_i+2}{a_i+1}\right)$$

where,

$$\delta_i(a_1, \ldots, a_n) = \begin{cases} 1 & \text{if } A \text{ covers } \bigcup_{a_n=n}^{3n+1} (a_1, \ldots, a_n) \\ 0 & \text{otherwise.} \end{cases}$$

Then the measure μ is defined by,

$$\mu(A) = \lim_{n\to\infty} \mu_n(A).$$

Note that provided $m \leqslant n$, since the Y_n are nested sets (see (8)),

$$\mu_n([0,\ 1]) = \mu_n(Y_m) = \mu_n(Y_{m-1})$$

and,

$$\begin{aligned}\mu_n(Y_m) &= (\log 3)^{1-n}\prod_{i=1}^{n-1}\sum_{a_i=1}^{3j+1}\log\left(\frac{a_i+2}{a_i+1}\right)\\ &= (\log 3)^{1-n}\prod_{i=1}^{n-1}\log\left(\frac{3\mathrm{i}+3}{\mathrm{i}+1}\right)\\ &= (\log 3)^{1-n}(\log 3)^{n-1}\\ &= 1.\end{aligned}$$

Thus, the total measure $\mu([0,\ 1]) = \mu(Y_m) = 1$, which is finite, positive and independent of m. Then as Y is defined as $\cap_{m=1}^{\infty} Y_m = \lim_{m\to\infty} Y_m$, then $\mu(Y) = 1 > 0$. Thus, the condition that is required in the final step to reach (10) is met. Then it remains to show that (9) is satisfied for any interval, I, in any cover, $\bar{Y}$, of Y. The intervals of cover $\bar{Y}$ will first be reconstructed slightly to simplify the analysis. The aim of the reconstruction is so that for any I, an interval of Y_m can be found, for some m, that covers I, such that the endpoints of I lie in distinct intervals of Y_{m+1}.

As $[0, 1]$ is a compact space, any cover has a finite subcover, so it will be assumed that $\bar{Y}$ is finite. This will have no effect on the infimum of (4). Now the size of the intervals in $\bar{Y}$ tend to zero as $n \to \infty$, so the intervals of $\bar{Y}$ (that are now finite in number), can be extended an arbitrarily small amount, such that the endpoints of these intervals lie outside Y_n, for sufficiently large n. The sum in (4) in then increased an arbitrarily small amount, having negligible effect on the infimum. The intervals of $\bar{Y}$ must then cover Y_n. Then reduce the intervals of $\bar{Y}$ until their endpoints meet those of Y_n. This series of operations has thus simplified the structure of $\bar{Y}$ with respect to Y_n.

For any $I \in \bar{Y}$, let m be the largest integer such that the interval $\cup_{a_m=m}^{3m+1}(a_1, \ldots, a_m)$ completely covers I. The construction in the preceding paragraph has maximized m. For each $a_m \in \{m, m+1, \ldots, 3m+1\}$, there will be a distinct interval $\cup_{a_{m+1}=m+1}^{3m+4}(a_1, \ldots, a_{m+1})$ of Y_{m+1} nested inside $\cup_{a_m=m}^{3m+1}(a_1, \ldots, a_m)$ of Y_m. After the reconstruction above, the endpoints of I lie in distinct intervals of Y_{m+1}. These two intervals can be labelled as s, t according to the a_m values the intervals take. Then $s, t \in \{m, m+1, \ldots, 3m+1\}$, with $s < t$. An upper bound on the left-hand side of (9) can obtained as follows. If $N > m$ then,

$$\begin{aligned}\mu_N(I) &\leqslant \sum_{a_m=s}^{a_m=t}\mu_N((a_1, a_2, \ldots, a_m))\\ &= (\log 3)^{1-N}\left[\prod_{i=1}^{m-1}\log\left(\frac{a_i+2}{a_i+1}\right)\right]\log\left(\frac{t+2}{s+1}\right)(\log 3)^{N-m-1}\\ &= (\log 3)^{-m}\log\left(\frac{t+2}{s+1}\right)\prod_{i=1}^{m-1}\log\left(\frac{a_i+2}{a_i+1}\right)\end{aligned}$$

which gives,

$$\mu(I) \leqslant (\log 3)^{-m}\log\left(\frac{t+2}{s+1}\right)\prod_{i=1}^{m-1}\log\left(\frac{a_i+2}{a_i+1}\right).$$

Then as $m \leqslant s,\ t \leqslant 3m+1$,

$$\mu(I) \leqslant (\log 3)^{-m}\log\left(\frac{3m+1+2}{m+1}\right)\prod_{i=1}^{m-1}\log\left(\frac{a_i+2}{a_i+1}\right)$$

so,

$$\mu(I) \leqslant (\log 3)^{1-m} \prod_{i=1}^{m-1} \log\left(\frac{a_i+2}{a_i+1}\right).$$

Finally, note that, $\log\left(\frac{a_i+2}{a_i+1}\right) \leqslant \frac{1}{a_i+1}$, so,

$$\mu(I) \leqslant \frac{1}{(\log 3)^{m-1}} \prod_{i=1}^{m-1} \left(\frac{1}{a_i+1}\right). \tag{16}$$

To obtain the right-hand side of (9) note that,

$$\begin{aligned} |I|^{\frac{1}{2}} &\geqslant \left| \bigcup_{a_{m+1}=3m+2}^{\infty} (a_1, \ldots, a_{m-1}, t, a_{m+1} \right|^{\frac{1}{2}} \\ &\geqslant \prod_{i=1}^{m-1} \left(\frac{1}{a_1+1}\right) \frac{1}{t+1} \left| \int_{a_{m+1}=3m+2}^{\infty} \frac{\mathrm{d}x}{(x+1)^2} \right|^{\frac{1}{2}} \ldots\ldots \text{(using (14))} \\ &\geqslant \prod_{i=1}^{m-1} \left(\frac{1}{a_i+1}\right) \frac{1}{3m+2} \frac{1}{\sqrt{3m+3}} \end{aligned}$$

so,

$$|I|^{\frac{1}{2}} \geqslant \frac{1}{(3m+3)^{\frac{3}{2}}} \prod_{i=1}^{m-1} \left(\frac{1}{a_i+1}\right).$$

In comparison with (16), this gives $\mu(I) \leqslant |I|^{\frac{1}{2}}$. Note that although this is only true while $m \geqslant 91$, for $m < 91$, $\mu(I)A|I|^{\frac{1}{2}}$, where $A = (3(90)+3)^{\frac{3}{2}}$ and (9) is obtained. Thus, the proof of theorem 1 is complete. □

6. Proving corollary 1.1

From theorem 1, the set of points $\alpha = [a_1, a_2, \ldots]$ such that $\lim_{n\to\infty} \frac{1}{n} \prod_{i=1}^{n} a_i = \infty$ has Hausdorff dimension $\frac{1}{2}$. From the recursive relation $q_n = a_n q_{n-1} + q_{n-2}$ given in (3), it follows that if $\prod_{i=1}^{n} a_i^{\frac{1}{n}}$ diverges, then $q_n^{\frac{1}{n}}$ certainly will. Thus, the set of theorem 1 (from here on referred to as X) is a subset of that considered in corollary 1.1 (referred to as $\bar{X}$). Thus, $\bar{X}$ automatically has a Hausdorff dimension with lower bound $\frac{1}{2}$. Essentially section 5 still applies. To complete the proof, section 4 needs to be modified slightly. So in a similar manner to (7), consider the set of points that satisfy,

$$\sum_{j=1}^{n} 2^j \prod_{i=1}^{j} a_i > \kappa^n. \tag{17}$$

This set includes $\bar{X}$ as a subset. This can be seen with the aid of induction. First assume,

$$2^n \prod_{i=1}^{n} a_i > q_n$$

to be true for n. Then,

$$2^{n+1} \prod_{i=1}^{n+1} a_i = 2a_{n+1} 2^n \prod_{i=1}^{n} a_i > 2a_{n+1} q_n > q_{n+1}$$

so as this is true for $n = 1$, it is also true for any n. Similarly, assume that for some value n,

$$\sum_{j=1}^{n} 2^j \prod_{i=1}^{j} a_i > q_n$$

then,

$$\sum_{j=1}^{n+1} 2^j \prod_{i=1}^{j} a_i = \sum_{j=1}^{n} 2^j \prod_{i=1}^{j} a_i + 2^{n+1} \prod_{i=1}^{n+1} a_i > q_n + 2^{n+1} \prod_{i=1}^{n+1} a_i > q_n + q_{n+1} > q_{n+1}.$$

The inequality is true for $n = 1$, so therefore also true for all n.

Thus, when q_n diverges, the sum certainly does, and the set considered in (17) covers $\bar{X}$. To prove the corollary, the method is the same as that of section 4, the following bound being obtained, in much the same way that (15) was,

$$H_n^s(\bar{X}) \leqslant \sum_{i=1}^{n} \frac{i^{\frac{3}{2}}}{\sqrt{2\pi}} \left(\frac{4e\zeta(2s)\log\kappa}{(2s-1)\kappa^{2s-1}} \right)^i .$$

Note that,

$$\frac{4e\zeta(2s)\log\kappa}{(2s-1)\kappa^{2s-1}} < 1$$

can be satisfied for any $s > \frac{1}{2}$, provided κ is sufficiently large. So by the ratio test convergence can be seen to occur as $n \to \infty$. Then if $H^s(\bar{X})$ converges, by (5), it must converge to zero. Thus, $\frac{1}{2}$ becomes an upper bound on the Hausdorff dimension $D(\bar{X})$, and so the corollary is complete. □

7. Conclusion

The Hausdorff dimension of $X(\kappa)$ has been determined exactly as $\kappa \to \infty$. It would be interesting to find $D(X(\kappa))$ exactly, for finite values of κ. From [5], the following result may be found,

$$\lim_{n\to\infty} \prod_{i=1}^{n} a_i^{\frac{1}{n}} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{2k+k^2}\right)^{\frac{\log\kappa}{\log 2}}$$

for all $\omega = [a_1, a_2, \ldots] \in [0, 1]$, bar a set of Lebegsue measure zero. Thus, for $\kappa \leqslant \prod_{k=1}^{\infty} \left(1 + \frac{1}{2k+k^2}\right)^{\frac{\log\kappa}{\log 2}}$ it is certainly true that $D(X(\kappa)) = 1$. Similarly, for all $\omega = [a_1, a_2, \ldots] \in [0, 1]$, bar a set of measure zero, it is true that,

$$\lim_{n\to\infty} q_n^{\frac{1}{n}} = \mathrm{e}^{\frac{\pi^2}{12\log 2}}.$$

So if $\bar{X}(\kappa)$ denotes the corresponding set to corollary 1.1, $D(\bar{X}(\kappa)) = 1$ for $\kappa < \mathrm{e}^{\frac{\pi^2}{12\log 2}}$, which suggests that $D(\bar{X}(\kappa)) \neq D(X(\kappa))$ when κ is finite. However, it seems likely that tighter bounds than those used in (14) would be required.

These studies arose during the studies into the level spacing distributions of harmonic oscillators of dimension two (see [6]). It was found that if ω was the ratio of the frequencies of the oscillations, where the condition of theorem 1 was satisfied, then a delta function was obtained for the spacing distribution. Note that the set in question is dense, meaning the level spacing distribution is unstable with respect to the frequency ratio.

Similar results can be obtained using a result of Ramanujan for the system of a particle trapped in a two-dimensional box.

Lemma 7.1. (*Ramanujan*) Let $Z(k)$ note the number of positive integers $m \leqslant k$ that can be expressed as the sum of two squares. Then,

$$\lim_{m\to\infty} \frac{Z(k)\sqrt{\ln k}}{k} = \sqrt{\frac{1}{2} \prod_{r\equiv 3(\mathrm{mod}4)} \left(1 - \frac{1}{r^2}\right)^{-1}}$$

Proof. See Hardy's 'Ramanujan' [10]. □

This was used to obtain the following result

Theorem 2. Consider the quantized system of a particle trapped in a two-dimensional box, such that the lengths of the sides of the box have ratio $\alpha = [a_0, a_1, a_2, \ldots]$. Then the level spacing distribution approaches $\delta(s)$ infinitely often if the a_i have a subsequence b_i such that

$$\lim_{n\to\infty} \frac{\ln b_{n+1}}{(a_1 \ldots a_n)^4} \to \infty.$$

Proof. See appendix. □

This, as in the previous system, is an example of a dense set of ratios that give non-generic behaviour. From [7], the generic distribution will be that of a Poisson process. It would be worth investigating whether the set of theorem 2 has a positive Hausdorff dimension.

In both the two-dimensional harmonic oscillator and the two-dimensional box, the results have been about the non-generic behaviour of their level spacing distribution, $P(s)$, under certain conditions, in particular that their distributions are delta functions. However, the average value of $\delta(s)$ is zero, whereas the average value of the level spacing distribution is defined to be unity, which appears to be a contradiction. Now $P(s)$ is defined as $\lim_{U\to\infty} P(s, U)$, where $P(s, U)$ is the spacing distribution defined on spacings corresponding to energy levels under an upper bound U (see [6]). For all finite values of U the distribution $P(s, U)$ has an average of unity. This gives the following inequality,

$$\lim_{U\to\infty} \langle P(s, U)\rangle \neq \left\langle \lim_{U\to\infty} P(s, U)\right\rangle$$

where the brackets indicate the average.

The level spacing distribution is also defined for positive spacings only, whereas the delta function suggests that most spacings are zero; another apparent contradiction. However, the delta function merely indicates that for any $\epsilon > 0$, the proportion of spacings $< \epsilon$ tends to unity as $U \to \infty$.

Although only two physical systems were looked at, the results obtained here suggest similar behaviour for level spacing distributions in general. Finding other systems with similar properties would be a worthwhile exercise.

It was interesting to see the areas of quantum mechanics, number theory and Hausdorff dimension interact in this unexpected way.

Acknowledgment

The author would like to thank Sandy Davie for numerous helpful comments.

Appendix

To obtain the level spacing distribution, $P(s)$, the quantized energy values are first normalized, so that the average separation between consecutive levels is unity. Denoting the normalized levels by $U_1 \leqslant U_2 \leqslant \ldots \leqslant U_i \leqslant U_{i+1} \leqslant \ldots$ (where degeneracy is represented by repetitions), first define $P(s, U)$ via,

$$P(s, U)\,\mathrm{d}s = \frac{1}{U}|\{U_i < U | U_{i+1} - U_i \in (s, s + \mathrm{d}s)\}|$$

where the vertical bars denote the size of the set, then $P(s) = \lim_{U\to\infty} P(s, U)$. For the system in question, the claim is that $P(s, U)$ approaches $\delta(s)$ infinitely often. More specifically, a diverging sequence $V_n = V_n(b_n)$ will be constructed such that $\lim_{n\to\infty} P(s, V_n) \approx \delta(s)$.

For the two-dimensional box (see [7]), the values U_i arise from,

$$U = \frac{\pi}{4}(\alpha m_1^2 + \alpha^{-1} m_2^2)$$

as m_1 and m_2 vary over the positive integers. For any n such that $a_{n+1} (= b_{n'})$ is a member of the subsequence $\{b_i\}$, consider the values $W_1 \leqslant W_2 \leqslant \ldots \leqslant W_i \leqslant W_{i+1} \leqslant \ldots$ arising from

$$W = \frac{\pi}{4}\left(\frac{p_n}{q_n} m_1^2 + \frac{q_n}{p_n} m_2^2\right) \Leftrightarrow \frac{4W p_n q_n}{\pi} = (p_n m_1)^2 + (q_n m_2)^2$$

as m_1 and m_2 vary over the positive integers, where p_n/q_n is the nth convergent to α. Then using lemma 6.1, the number of distinct values $W_i < W$ that occur will be

$$\sqrt{\frac{1}{2}\prod_{r\equiv 3(\mathrm{mod} 4)} (1 - r^{-2})^{-1}}\,\frac{4W p_n q_n}{\pi\sqrt{\ln(W p_n q_n)}}.$$

As the total number of $W_i < W$, is W (as the average spacing is unity), the proportion of distinct values is,

$$\sqrt{\frac{8\prod_{r\equiv 3(\mathrm{mod} 4)} (1 - r^{-2})^{-1} p_n^2 q_n^2}{\pi^2 \ln(W p_n q_m)}}.$$

If $P(s, W)$ denotes the distribution arising from the W_i spacings, the value above will be the proportion of non-zero spacings. If this quantity tends zero, then $P(s, W)$ tends to $\delta(s)$, i.e. all spacing are essentially zero. Equivalently, $\frac{\ln(W q_n)}{q_n^4}$ is required to diverge.

Now as $\alpha \approx \frac{p_n}{q_n}$, then $W_i \approx U_i$, so to each zero spacing of $P(s, W)$, there will correspond a small spacing that contributes to $P(s, U)$. A condition on a_{n+1} shall be found such that when $U = V_{n'}$, $\frac{\ln(W q_n)}{q_n^4}$ diverges and the small spacings are all bounded by some $\epsilon > 0$. Now,

$$|U_i - W_i| = \frac{\pi}{4}\left|m_1^2\left(\alpha - \frac{p_n}{q_n}\right) + m_2^2\left(\alpha^{-1} - \frac{q_n}{p_n}\right)\right|$$

but from [1], $\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q'_{n+1}}$, giving,

$$|U_i - W_i| = \frac{\pi}{4}\left|\frac{m_1^2(-1)^n}{q_n q'_{n+1}} - \frac{m_2^2(-1)^n}{p_n p'_{n+1}}\right| \leqslant \frac{\pi}{4}\frac{1}{q_n q_{n+1}}|m_1^2 + \alpha^{-1} m_2^2| \leqslant \frac{U_i}{\alpha q_n q_{n+1}}$$

which gives an upper bound on the small spacings. So if $V_n = \epsilon\alpha q_n q_{n+1}$, where $U = V_n$, the small spacings are bounded by ε. Then if $\frac{\ln(U q_n)}{q_n^4}$ diverges, all spacings will be bounded, and so $P(s, U)$ can be made arbitrarily close to $\delta(s)$, as ϵ can be arbitrarily small. This completes the proof. □

References

[1] Hardy G H and Wright E M 1938 *An Introduction to the Theory of Numbers* (Oxford: Oxford University Press)
[2] Khovanskii 1963 *The Applications of Continued Fractions and their Generalisations* (Library of Applied Analysis)
[3] Falconer K J 1990 *Fractal Geometry* (New York: Wiley)
[4] Edgar G A 1990 *Measure, Topology and Fractal Geometry* (Cambridge: Cambridge University Press)
[5] Cornfeld I P, Fomin S V and Sinai Ya G 1982 *Ergodic Theory* (Berlin: Springer)
[6] Greenman C D 1996 The generic spacing distribution of the two dimensional harmonic oscillator *J. Phys. A: Math. Gen.* **29** 4065–81
[7] Berry M V and Tabor M 1977 Level clustering in the regular spectrum *Proc. R. Soc.* A **356** 375–94
[8] Bohigas O, Giannoni M J and Schmit C 1984 Characterization of chaotic quantum spectra and universality of level fluctuation laws *Phys. Rev. Lett.* **52** 1–4
[9] Boas M L 1996 *Mathematical Methods in the Physical Sciences* (New York: Wiley)
[10] Hardy G H 1940 *Ramanujan* (Cambridge: Cambridge University Press)